

A non-equilibrium quantum Landauer principle

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Using the operational framework of completely positive, trace preserving operations and thermodynamic fluctuation relations, we derive a lower bound for the heat exchange in a Landauer erasure process on a quantum system. Our bound comes from a non-phenomenological derivation of the Landauer principle which holds for generic non-equilibrium dynamics. Furthermore the bound depends on the non-unitality of dynamics, giving it a physical significance. We apply our framework to the model of a spin-1/2 system coupled to an interacting spin chain at finite temperature.

Introduction.— The most convincing evidence that information is indeed physical is provided by Landauer's principle [1]. It states that the logically-irreversible erasure of information carried by a physical system comes at the expense of heat dissipation to the environment. Stated equivalently, the principle provides a direct link between the domains of information theory and thermodynamics. The deep consequences of Landauer's principle were instrumental for Bennett to attach a minimum entropy production to the logically irreversible procedure of erasure [2, 3], thus operating an information-theoretical exorcism of Maxwell demon and recognising that computation can be done reversibly, in principle, requiring no heat production.

Turning to quantum systems, it is surprising that very few papers have a clear operational framework which is suitable to understand the emergence of Landauer's principle from the underlying microscopic equations. In a recent work [4], Reeb and Wolf use techniques from quantum statistical mechanics to prove that a finite-size environment can provide a tighter bound to the quantity of heat generated in an erasure process. The usual Landauer bound

$$\beta \langle Q \rangle \geq \Delta S \quad (1)$$

is retrieved when the thermodynamic limit is taken in the environment. In Eq. (1), $\langle Q \rangle$ is the average heat exchange with the bath and ΔS is the information theoretic entropy change. The finite size corrections to Eq. (1) proposed in Ref. [4] are, in some sense, suggestive of intrinsic non-equilibrium dynamics which one would expect away from the thermodynamic limit.

One way to describe the thermodynamics of systems where thermal and quantum fluctuations cannot be ignored is to consider the so called work and heat fluctuation relations [5–8]. The fluctuation relations, extended to the quantum mechanical domain [9, 10] are a promising route to understand the thermodynamics of small quantum systems that are operating under non-equilibrium conditions. Crucially, recent work has demonstrated that the fluctuation formalism is a tangible route for the experimental exploration of quantum thermodynamics [11–13].

In this Letter we bring together the tools of open quantum systems, non-equilibrium statistical mechanics and quantum

information theory to derive a non-phenomenological lower bound for heat generated in an erasure process. We begin by recasting the erasure protocol given in Ref. [4] from the point of view of fluctuation relations. Extension of the fluctuation relation formalism to the open quantum-system framework is leads to difficulties unless fairly restrictive assumptions are made [14, 15]. A series of papers have attempted to derive fluctuation relations from the operational viewpoint, employing the full machinery of completely positive, trace preserving operations [16–20], which are ubiquitous in quantum information. It was found that fluctuation relations of the standard form hold if the open system evolution is unital [21].

We construct a distribution of heat generated for an erasure protocol in the environment that is in touch with a quantum system. We show that the non-unitality of a given open-system dynamics can lead to a tighter bound on the average heat exchanged across the process than previously known bounds, thus significantly improving the framework putting forward corrections to the Landauer's principle due to finite-size environment. In addition, and more importantly, the methodology developed here paves the way to the tantalising possibility of analysing the thermodynamics of computation under non-equilibrium conditions. Our work brings together concepts from several disciplines of physics and constitutes a promising route for the construction of a formal platform for exploring the efficiency and limitations of small-scale thermodevices operating at and well within the quantum mechanical domain [22].

Erasure protocol.— Starting from the analysis put forward in Ref. [4], the starting point of our investigation is embodied by a general erasure protocol that satisfies the following set of pre-requisites:

1. A system \mathcal{S} , whose information content we want to erase, is subjected to an environment-aided erasure protocol. We call $\hat{H}_{\mathcal{S}}$ the free Hamiltonian of the system.
2. We label such environment as \mathcal{E} and assume that the initial total \mathcal{SE} state is fully uncorrelated, i.e. $\hat{\rho}_{\mathcal{SE}} = \hat{\rho}_{\mathcal{S}} \otimes \hat{\rho}_{\mathcal{E}}$.
3. The environment is initially in the Gibbs state $\hat{\rho}_{\mathcal{E}} = e^{-\beta \hat{H}_{\mathcal{E}}} / Z_{\mathcal{E}}$, with Hamiltonian $\hat{H}_{\mathcal{E}} = \sum_m E_m |r_m\rangle \langle r_m|$, the

inverse temperature β , and the partition function $Z_E = \text{tr}[e^{-\beta \hat{H}_E}]$.

4. The system and environment interact via a perfectly unitary mechanism described by the time propagator \hat{U} generated by the total Hamiltonian $\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{SE}$.

These points constitute a non-restrictive set of *rules* for the erasure protocol to be performed.

Thermodynamics of the environment.— We describe the erasure process as the protocol that, starting from the joint SE initial state, generates the following reduced state of the environment only

$$\begin{aligned} \hat{\rho}'_E &= \text{tr}_S[\hat{U}(\hat{\rho}_S \otimes \hat{\rho}_E)\hat{U}^\dagger] \\ &= \sum_{j,k} \lambda_j \langle s_k | \hat{U} | s_j \rangle \hat{\rho}_E \langle s_j | \hat{U}^\dagger | s_k \rangle \\ &= \sum_l \hat{A}_l \hat{\rho}_E \hat{A}_l^\dagger, \end{aligned} \quad (2)$$

where we used the eigenvalues $\{\lambda_j\}$ and corresponding eigenstates $\{|s_j\rangle\}$ of $\hat{\rho}_S$, which is thus decomposed as $\hat{\rho}_S = \sum_j \lambda_j |s_j\rangle \langle s_j|$. Moreover, we have introduced the set of operators for the environment $\hat{A}_{l=jk} = \sqrt{\lambda_j} \langle s_k | \hat{U} | s_j \rangle$. It is worth stressing that Eq. (2) does not embody a map, but an operation. In fact, we can vary $\hat{\rho}_S$, thus changing the form of the \hat{A}_l 's while keeping $\hat{\rho}_E$ fixed. The operation is trace-preserving when $\sum_l \hat{A}_l^\dagger \hat{A}_l = \mathbb{1}_E$ and it is unital if and only if $\sum_l \hat{A}_l \hat{A}_l^\dagger = \mathbb{1}_E$. While the trace-preserving condition can be shown to be rigorously satisfied, a generic quantum operation is not unital.

In analogy to work distribution, we now define the heat distribution for the environment [23] as

$$P(\mathbf{Q}) = \sum_{l,m,n} \langle r_n | \hat{A}_l | r_m \rangle (\hat{\rho}_E)_{mm} \langle r_m | \hat{A}_l^\dagger | r_n \rangle \delta(\mathbf{Q} - E_{nm})$$

with $(\hat{\rho}_E)_{nm} = \langle r_n | \hat{\rho}_E | r_m \rangle$ is the matrix element of the environmental initial state in the basis of its eigenstates and $E_{nm} = E_n - E_m$. The first moment of the heat distribution is the average heat, which can be written as

$$\begin{aligned} \langle \mathbf{Q} \rangle &= \int \mathbf{Q} P(\mathbf{Q}) d\mathbf{Q} \\ &= \sum_n E_n (\hat{\rho}'_E)_{nn} - \sum_m E_m \sum_l \text{tr}[\hat{A}_l \hat{\rho}_E \hat{A}_l^\dagger] \\ &= \text{tr}[\hat{H}_E \hat{\rho}'_E] - \text{tr}[\hat{H}_E \hat{\rho}_E]. \end{aligned} \quad (3)$$

The distribution of heat contains much more information than just the first moment. For instance, Jarzynski has found an important equality using the work distribution, which relates the average exponentiated work to the equilibrium free energy. In the same spirit, we now evaluate the average exponentiated

heat to derive a heat fluctuation relation:

$$\begin{aligned} \langle e^{-\beta \mathbf{Q}} \rangle &= \int e^{-\beta \mathbf{Q}} d\mathbf{Q} P(\mathbf{Q}) \\ &= \sum_{l,m,n} \langle r_n | \hat{A}_l | r_m \rangle \langle r_m | \hat{A}_l^\dagger | r_n \rangle (\hat{\rho}_E)_{mm} e^{-\beta E_{nm}} \\ &= \sum_{l,m,n} \langle r_m | \hat{A}_l^\dagger | r_n \rangle \langle r_n | \hat{A}_l | r_m \rangle \frac{e^{-\beta E_n}}{Z_E} \\ &= \sum_{l,m} \langle r_m | \hat{A}_l^\dagger \sum_n \frac{e^{-\beta r_n}}{Z_E} | r_n \rangle \langle r_n | \hat{A}_l | r_m \rangle \\ &= \sum_l \text{tr}[\hat{A}_l^\dagger \hat{\rho}_E \hat{A}_l] = \text{tr}[\hat{\mathbf{A}} \hat{\rho}_E], \end{aligned}$$

with $\hat{\mathbf{A}} = \sum_l \hat{A}_l \hat{A}_l^\dagger$.

Care should be used in order to interpret the above expression as a fluctuation relation. Strictly speaking it is a fluctuation relation only for unital or bi-stochastic processes [16, 18–20]. On the contrary, the process at hand is surely not unital, as the erasure of S would inevitably perturb a hypothetically prepared maximally mixed state of the environment (i.e., a Gibbs state at infinite temperature). Unital processes will give $\hat{\mathbf{A}} = \mathbb{1}_E$. Eq. (4) provides an interpretation of the non-unitality of the operation under scrutiny as the average exponentiated heat exchanged with the environment.

This result is one of the key steps for the formulation of our bound. Let us expand the operator $\hat{\mathbf{A}}$ in terms of the initial states of both system and environment under the action of the unitary process \hat{U} . We have

$$\langle e^{-\beta \mathbf{Q}} \rangle = \sum_{j,k} \text{tr}[\lambda_j \langle s_j | \hat{U}^\dagger | s_k \rangle \hat{\rho}_E \langle s_k | \hat{U} | s_j \rangle] \quad (4)$$

$$= \text{tr}[\hat{\rho}_S \otimes \mathbb{1}_E \hat{U}^\dagger \mathbb{1}_S \otimes \hat{\rho}_E \hat{U}]. \quad (5)$$

Now we use the Jensen inequality $\langle f(x) \rangle \geq f(\langle x \rangle)$, which holds for any convex function $f(x)$, to get $e^{-\beta \langle \mathbf{Q} \rangle} \leq \langle e^{-\beta \mathbf{Q}} \rangle = \text{tr}[\hat{\rho}_S \otimes \mathbb{1}_E \hat{U}^\dagger \mathbb{1}_S \otimes \hat{\rho}_E \hat{U}]$, yielding the final bound $\beta \langle \mathbf{Q} \rangle \geq \mathcal{B}_Q$ with

$$\mathcal{B}_Q = -\ln(\text{tr}[\hat{\rho}_S \otimes \mathbb{1}_E \hat{U}^\dagger \mathbb{1}_S \otimes \hat{\rho}_E \hat{U}]). \quad (6)$$

This inequality represents the key contribution of our analysis. The bound on the average heat exchanged with the environment during the erasure process has been derived from fluctuation-relation premises and is heavily reliant on the non-unital nature of the operation being implemented on the environment. This extra term, that only exists for non-unital operations (on the environment), was discussed in [16, 18–20] and here we give it a physical meaning.

Usage and interpretation of the bound.— Let us discuss how this bound can be of operational use. Suppose we are interested in buying an erasure machine which comes in a closed box with secret $\hat{\rho}_E$ and \hat{U} . The company that makes the erasure machine tells us the output state corresponding to the maximally mixed input state of the system when it jointly evolves under \hat{U}^\dagger with $\hat{\rho}_E$: $\hat{M}_S = \text{tr}_E[\hat{U}^\dagger \frac{\mathbb{1}_S}{d_S} \otimes \hat{\rho}_E \hat{U}]$. Rear-

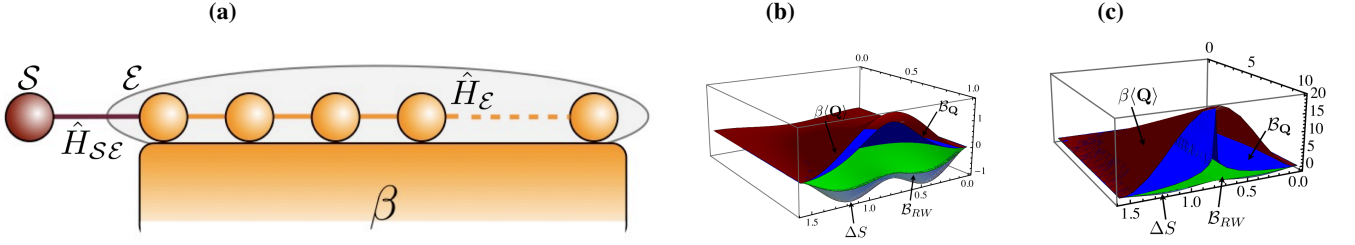


FIG. 1. **(a)** Schematic representing the system under consideration. **(b)** Comparison between $\beta\langle Q \rangle$, the bound \mathcal{B}_Q derived in Eq. (6), and the one found in Ref. [4] for a spin-1/2 particle interacting for a dimensionless time Jt with a single-spin environment at inverse temperature $\beta = 1$. We also plot the change in entropy ΔS . All the quantities are studied against the initial preparation $\alpha|1\rangle_S + \sqrt{1-\alpha^2}|0\rangle_S$ ($\alpha \in \mathbb{R}$) of the system state. **(c)** Analogous comparison as in panel **(b)**, but performed against the environmental temperature and for the system being prepared in the pure state $|1\rangle_S$.

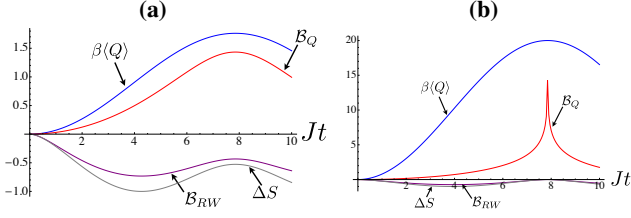


FIG. 2. Plot of the key quantities in our analysis and comparison with \mathcal{B}_{RW} for $B_0/J = B/J = 1$ and $J_0/J = 0.1$ for $\beta = 1$ [panel **(a)**] and $\beta = 10$ [panel **(b)**]. In both panels we have taken $\alpha = 1$.

ranging Eq. (6) we get

$$\mathcal{B}_Q = -\ln(\text{tr}[\hat{\rho}_S \hat{M}_S]) - \ln(d_S). \quad (7)$$

With the extra knowledge of \hat{M}_S , the bound becomes useful for operational purposes. It is worth noting that our bound above is at times tighter than the Reeb-Wolf bound \mathcal{B}_{RW} . We will show this in detail in the next section with a physically motivated example of a finite-size environment. Before we do that, let us mention are instances where our bound is worse than the standard Landauer's bound (and therefore also worse than \mathcal{B}_{RW}). For instance, consider when ρ_S is chosen to be the maximally mixed state. Then the change in the entropy of the system is positive, while $\mathcal{B}_Q = 0$. Nevertheless, this tells us that for any unital processes on the environment, heat, on average, will always flow from the system to the environment. It is also worth to point out that our result agrees with the fluctuation relation derived by Talkner *et al.* in [24] when our system begins in a thermal state.

Physical model.— We consider a spin-1/2 system, whose logical states are labelled as $|0\rangle_S$ and $|1\rangle_S$, interacting with an environment embodied by an interacting spin chain of N elements that we assume in contact with a thermostat at inverse temperature β . The system is sketched for illustrative purposes in Fig. 1 **(a)**. The Hamiltonian of the environment is given by the isotropic XX model

$$\hat{H}_E = J \sum_{j=1}^{N-1} (\hat{\sigma}_x^j \hat{\sigma}_x^{j+1} + \hat{\sigma}_y^j \hat{\sigma}_y^{j+1}) + B \sum_{j=1}^N \hat{\sigma}_z^j \quad (8)$$

with J the inter-spin coupling strength, B the strength of the coupling between each spin and a homogeneous external magnetic field and $\hat{\sigma}_k^j$ is the k -Pauli spin operator ($k = x, y, z$) for particle $j = 1, \dots, N$. The system spin evolves freely according to the Hamiltonian $B_0 \hat{\sigma}_z^S = B_0(|1\rangle\langle 1|_S - |0\rangle\langle 0|_S)$ and is coupled to the environment through element 1 of the chain according to the model

$$\hat{H}_{SE} = J_0 \sum_{k=x,y} \hat{\sigma}_k^S \hat{\sigma}_k^1. \quad (9)$$

In line with the general requirements of the erasure protocol highlighted before, we will assume that the environmental system is prepared in the equilibrium state $\rho_E = e^{-\beta \hat{H}_E} / Z_E$. For $B_0 = B$ and $J_0 = J$, the dynamics resulting from the model $\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{SE}$ is then equivalent to that of a time-dependent generalised amplitude damping channel described by the Kraus operator

$$\begin{aligned} \hat{A}_0 &= \sqrt{p} \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{A}_1 = \sqrt{p} \begin{pmatrix} 0 & 0 \\ \sqrt{1-\phi^2} & 0 \end{pmatrix}, \\ \hat{A}_2 &= \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix}, \quad \hat{A}_3 = \sqrt{1-p} \begin{pmatrix} 0 & \sqrt{1-\phi^2} \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (10)$$

with $\phi = {}_S\langle 1|_E \langle 0 \dots 0| e^{-i\hat{H}t} |0 \dots 0\rangle_E |1\rangle_S$ [26] and $p \in [0, 1]$ a probability whose value is linked to the equilibrium temperature of the environment. Needless to say, the explicit form

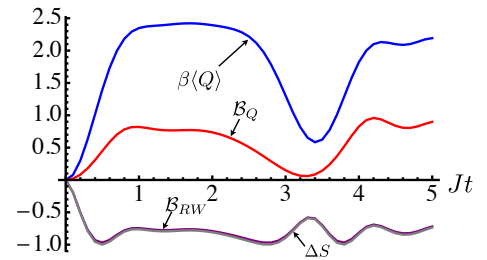


FIG. 3. Similarly to Fig. 2 **(a)** and **(b)**, we plot the key quantities of our study for $J_0/J = B_0/J = B/J = 1$, $\beta = 1$, $\alpha = 1$ and an environment of $N = 4$ elements. The curve showing the behaviour of ΔS is basically indistinguishable from the one for \mathcal{B}_{RW} .

of ϕ depends on the size of the environmental subsystem, i.e. on the number N of spins that compose it. In light of the overall non-unital nature of the generalised amplitude damping channel, this physical model embodies an example that is significant for our goals.

We allow us the freedom to prepare the system spin in any pure state $|\psi_i\rangle_S = \alpha|1\rangle_S + \sqrt{1-\alpha^2}|0\rangle_S$ with $\alpha \in \mathbb{R}$, for simplicity (the generality of our results is not affected by such assumptions, and the case of an arbitrarily mixed system state can be equally considered). In this case, the process at hand will make the entropy of the system increase. This implies that we are automatically excluding the cases where the state of the system is “reset”, which will be studied in future work. We start addressing the case of a single-spin environment, which provides a useful benchmark for our analysis, and then attack the case of a larger-size \mathcal{E} to investigate the scaling of our findings with the dimension of the environmental Hilbert space. In such simple, yet interesting case, we easily get $\phi = \cos(2Jt)$ and $p = e^{-\beta B}/Z_E = [1 + \tanh(\beta B)]/2$, which in turn lead us to

$$\beta\langle\mathbf{Q}\rangle = B\sin^2(2Jt)(2\alpha^2 + \tanh(\beta B) - 1), \quad (11)$$

$$\mathcal{B}_Q = -\ln[2(1-\phi^2)(\alpha^2 + p - 2p\alpha^2) + \phi^2] \quad (12)$$

with $\hat{\rho}'_S = \text{tr}_E[e^{-i\hat{H}t}|\psi_i\rangle\langle\psi_i|_S \otimes \hat{\rho}'_E e^{i\hat{H}t}]$ being the reduced state of the system S at time t . We now use the definition $\Delta S = S(\hat{\rho}_S) - S(\hat{\rho}'_S)$ with $S(\hat{\rho}) = -\text{tr}[\hat{\rho}\ln(\hat{\rho})]$ the von Neumann entropy of a generic state $\hat{\rho}$. As $S[|\psi_i\rangle\langle\psi_i|_S] = 0$, we have $\Delta S = -S[\hat{\rho}'_S]$. As this quantity turns out to be negative at all times, the relevant bound from Ref. [4] takes the form

$$\mathcal{B}_{RW} = \mathcal{R} - \sqrt{\mathcal{R}^2 + 2\mathcal{R}S[\hat{\rho}'_S]} \quad (13)$$

with $\mathcal{R} = \max_{0 < r < 0.5} r(1-r)\ln^2[(1-r)(d-1)/r]$ and $d = 2^{N-1}$. Although $S[\hat{\rho}'_S]$ can be computed analytically, its expression is too cumbersome to be reported here. A comparison among such quantities is made in Fig. 1 (b) and (c). The first of such plots shows that, at a set value of the environmental temperature, $\mathcal{B}_Q \geq \mathcal{B}_{RW}$ when $\alpha \simeq 1$, i.e. when the system preparation is close to the fully polarised state $|1\rangle_S$. This persists against temperature: in Fig. 1 (c) we present the behavior of $\beta\langle\mathbf{Q}\rangle$, \mathcal{B}_Q , and \mathcal{B}_{RW} against β , showing how at low temperature there is a time at which $(\beta\langle\mathbf{Q}\rangle - \mathcal{B}_Q) \rightarrow 0$, while the distance between the two bounds increases significantly. Needless to say, the non-unital nature of the process is not sufficient to guarantee a tighter bound than \mathcal{B}_{RW} , and there are regions in the (Jt, α, β) space where the dimension-dependent quantity proposed in Ref. [4] embodies a tighter bound to $\beta\langle\mathbf{Q}\rangle$ than \mathcal{B}_Q . Analogous conclusions can be reached for other choices of the parameters entering \hat{H} . For instance, in Fig. 2 we show the behaviour of the key quantities addressed in our study for $B_0/J = B/J = 1$ and $J_0/J = 0.1$ with $\beta = 1$ [panel (a)] and $\beta = 10$ [panel (b)], thus embodying the case of a system spin that is weakly coupled to the environment. As the bound \mathcal{B}_Q turns out to be always larger than \mathcal{B}_{RW} in both such cases,

these results suggest that non-Markovian effects induced by finite-size environments or strong $S\mathcal{E}$ coupling do not play a major role in this context.

Finally, it is worth addressing the case of larger environmental systems to study the effects of dimensionality, which we know are key in the Reeb and Wolf formulation [4]. To this end, we have considered the fully isotropic model for N growing from 1 to 6, finding results that are in agreement with the analysis presented above, albeit the interacting nature of the environmental subsystem makes the dynamics much richer than in the simple case of $N = 1$, as shown in Fig. 2 for the case of $N = 4$. Evidently, as the size of the environment grows ΔS and \mathcal{B}_{RW} become basically indistinguishable. While this study is sufficient to illustrate the key features of the model at hand, a more extensive analysis of the implications of criticality and the full assessment of size-scaling will be presented elsewhere.

Conclusions.— We have proposed a fluctuation-relation inspired framework for the addressing of Landauer’s erasure principle. Our approach provides a bound to the amount of heat exchanged with a finite-size environment during the erasing process that can be tighter than recently proposed modifications based on finite-size baths [4]. By linking it to the average exponentiated heat exchanged, our study provides a clear interpretation for the non-unitality of the erasure process, which acquires, in our study, the role of a resource. In fact, we have addressed the case of a generalised amplitude damping channel as the erasure mechanism for a single spin-1/2 system, showing that a better bound to the amount of exchanged heat can indeed be provided. Our study opens up a number of interesting avenues linked, for instance, to the study of the heat exchanges in microscopic models for system-environment interaction, the link to the emergence of non-Markovian features, design of quantum enhanced transport networks and the performance of thermodynamical cycles in quantum-inspired machines and motors [22, 27].

Acknowledgments. We are grateful to T. J. G. Apollaro, L. Céleri, G. De Chiara, L. Mazzola, D. Reeb, J. Anders, R. Sarthour, and V. Vedral for invaluable discussions and suggestions. We acknowledge financial support from the Alexander von Humboldt Foundation, the UK EPSRC (EP/G004579/1), the John Templeton Foundation (grant IDs 21806 and 43467), and the EU Collaborative Project TherMiQ (Grant Agreement 618074). This work was partially supported by the COST Action MP1209.

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